# Exact Complexiton Solutions of the (2+1)-Dimensional Burgers Equation

Yong Chen[a,c] and Qi Wang[b,c]

[a] Nonlinear Science Center, Ningbo University, Ningbo 315211, China
[b] Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China
[c] M. M. Key Lab, Chinese Academy of Sciences, Beijing 100080, China

Reprint requests to Y. C.; E-mail: chenyong@dlut.edu.cn

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Based on two different Riccati equations with different parameters, many new types of complexiton solutions to the (2+1)-dimensional Burgers equation are investigated. Such complexiton solutions obtained possess various combinations of trigonometric periodic and hyperbolic function solutions, various combinations of trigonometric periodic and rational function solutions, various combinations of hyperbolic and rational function solutions. – PACS numbers: 02.30.Ik, 05.45.Yv

*Key words:* Multiple Riccati Equations Rational Expansion Method; Complexiton Solutions; (2+1)-Dimensional Burgers Equation.

## 1. Introduction

The Riccati equation is used as subequation to find many algorithms for the construction of particular travelling solutions of a large number of nonlinear evolution equations [1 – 3]. Generally speaking, the various extensions and improvements of the Riccati subequation methods focus mainly on presenting a more general expansion ansatz. However, by the existing Riccati subequation methods the following complexiton solutions can not be obtained: combination of trigonometric periodic and hyperbolic function solutions, combination of trigonometric periodic and rational function solutions, combination of hyperbolic and rational function solutions. Recently we presented a Riccati equation rational expansion method [4] to obtain a series of travelling wave solutions including rational form solitary wave solutions, triangular periodic wave solutions and rational wave solutions. In the present paper, solutions of two different Riccati equations with different parameters are used as two variables in the components of a finite rational expansion. This method is named the multiple Riccati equations rational expansion (MRERE) method, in which we introduce a new ansatz in terms of a finite rational formal expansion:

$$U_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} \phi^{r_{j1}}(\xi) \psi^{r_{j2}}(\xi)}{(\mu_1 \phi(\xi) + \mu_2 \psi(\xi) + 1)^j}, \quad (1.1)$$

where $a^j_{r_{j1}r_{j2}}$, $\mu_1$ and $\mu_2$ ($r_{jn} = 1,2,\cdots,j$; $j = 0,1,\cdots,m_i$; $n = 1,2$; $i = 1,2,\cdots$) are constants to be determined later, and the new variables $\phi = \phi(\xi)$ and $\psi = \psi(\xi)$ satisfy two different Riccati equations, i. e.

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = h_1 + h_2\phi^2, \quad \frac{\mathrm{d}\psi}{\mathrm{d}\xi} = h_3 + h_4\psi^2, \quad (1.2)$$

where $h_1$, $h_2$, $h_3$ and $h_4$ are constants. So we can find many new types of complexiton solutions: various combinations of trigonometric periodic and hyperbolic function solutions, various combinations of trigonometric periodic and rational function solutions, various combinations of hyperbolic and rational function solutions, etc. We apply the MRERE method to the (2+1)-dimensional Burgers equation and find many new types of complexiton solutions, in the hope that they will lead to a deeper and more comprehensive understanding of the solution structures resulting from the (2+1)-dimensional Burgers equation.

## 2. Summary of the Multi-variable Riccati Equation Rational Expansion Method

In the following we would like to outline the main steps of our method:

*Step 1.* Given a system of polynomial PDEs with constant coefficients, with some physical fields $u_i(x,y,t)$ in three variables $x$, $y$, $t$,

$$\Delta\,(u_i, u_{it}, u_{ix}, u_{iy}, u_{itt}, u_{ixt}, u_{iyt}, u_{ixx}, u_{iyy}, u_{ixy}, \cdots) = 0, \quad (2.1)$$

using the wave transformation

$$u_i x,y,t) = U_i(\xi), \quad \xi = k(x+ly+\lambda t), \qquad (2.2)$$

where $k$, $l$ and $\lambda$ are constants to be determined later. Then the nonlinear partial differential system (2.1) is reduced to a nonlinear ordinary differential system:

$$\Theta(U_i, U_i', U_i'', \cdots) = 0. \qquad (2.3)$$

*Step 2.* We introduce a new ansatz in terms of finite rational formal expansion in the following form:

$$U_i(\xi) = a_{i0} + \sum_{j=1}^{m_i} \frac{\sum\limits_{r_{j1}+r_{j2}=j} a^j_{r_{j1}r_{j2}} \phi^{r_{j1}}(\xi)\psi^{r_{j2}}(\xi)}{(\mu_1\phi(\xi)+\mu_2\psi(\xi)+1)^j}, \qquad (2.4)$$

where $a^j_{r_{j1}r_{j2}}$, $\mu_1$ and $\mu_2$ ($r_{ji} = 1,2,\cdots,j$; $j = 1,2,\cdots,m_i$; $i = 1,2$) are constants to be determined later and the new variables $\phi = \phi(\xi)$ and $\psi = \psi(\xi)$ satisfy two different Riccati equation, i. e.

$$\frac{d\phi}{d\xi} = h_1 + h_2\phi^2, \quad \frac{d\psi}{d\xi} = h_3 + h_4\psi^2, \qquad (2.5)$$

where $h_1$, $h_2$, $h_3$ and $h_4$ are constants.

*Step 3.* Determine the $m_i$ of the rational formal polynomial solutions (2.4) by respectively balancing the highest nonlinear terms and the highest-order partial derivative terms in the given system equations (see [1–5] for details), and then give the formal solutions.

*Step 4.* Substitute (2.4) into (2.3) along with (2.5), and then set all coefficients of $\phi^p(\xi)\psi^q(\xi)(p = 0,1,2,\cdots; q = 0,1,2,\cdots)$ of the resulting system's numerator to zero to get an over-determined system of nonlinear algebraic equations with respect to $k$, $\mu_1$, $\mu_2$ and $a^j_{r_{j1}r_{j2}}$ ($r_{ji} = 1,2,\cdots$; $j = 1,2,\cdots,m_i$; $i = 1,2$).

*Step 5.* By solving the over-determined system of nonlinear algebraic equations using the symbolic computation system Maple, we end up with the explicit expressions for $k$, $\mu_1$, $\mu_2$, and $a^j_{r_{j1}r_{j2}}$ ($r_{ji} = 1,2,\cdots$; $j = 1,2,\cdots,m_i$; $i = 1,2$).

*Step 6.* According to (2.2) and (2.4), the conclusions in Step 5 and the general solutions of (2.5), which can be seen in the Appendix, we can obtain rational formal exact solutions of system (2.1), which include many new types of complexiton solutions.

*Remark 1:* The method proposed here is more general than the various existing methods [1–4] for finding exact solutions of nonlinear partial differential equations (PDEs). The appeal and success of the method lies in the fact that writing the exact solutions of a nonlinear equation as polynomials of $\phi$ and $\psi$, whose derivations are in closed form, the equation can be changed into a nonlinear system of algebraic equations. The system can be solved with the help of symbolic computation.

*Remark 2:* Due to using two different Riccati equations with different parameters as subequations, we can easily see that when $h_1 \neq h_3$ or $h_2 \neq h_4$, $\phi$ and $\psi$ satisfy different Riccati equations, so hyperbolic functions and triangular functions can appear in one solution at the same time. These solutions have not been obtained by any other Riccati equation expansion method or projective Riccati equation expansion method [1–5]. Clearly, there are many combinations, thus we omit them.

*Remark 3:* W. X. Ma [6] defined the complexiton solutions and found the complexiton solutions to the (Korteweg - de Vries (KdV) equation through its bilinear form. By means of a mapping relation, Lou et al. [7] found many new types of complexiton solutions. Although our method can not recover all complexiton solutions obtained by Ma's method and Lou's method, other new types of complexiton solutions can not be found by Ma's method and Lou's method. In particular, our method is a unified straightforward and pure algebraic algorithm to integrable and nonintegrable equations, which is implemented in a computer algebraic system. Of course, our method can also be extended to other integrable and nonintegrable systems.

## 3. Exact Complexiton Solutions of the (2+1)-Dimensional Burgers Equation

Let us consider the (2+1)-dimensional Burgers equation

$$\begin{aligned} &-u_t + uu_y + \alpha v u_x + \beta u_{yy} + \alpha\beta u_{xx} = 0, \\ &u_x - v_y = 0. \end{aligned} \qquad (3.1)$$

An equivalent form of the Burgers equation (3.1) is derived from the generalized Painlevé integrability classification in [8].

In order to get some families of rational form wave solutions to the (2+1)-dimensional Burgers equation, by considering the wave transformations $u(x,y,t) = U(\xi)$, $v(x,y,t) = V(\xi)$ and $\xi = k(x+ly+\lambda t)$, we change (3.1) to the form

$$\begin{aligned} &-\lambda U' + lUU' + \alpha VU' + \beta kl^2U'' \\ &\quad + \alpha\beta kU'' = 0, \\ &U' - lV' = 0. \end{aligned} \qquad (3.2)$$

For the (2+1)-dimensional Burgers equation, by balancing the highest nonlinear terms and the highest-order partial derivative terms in (3.2), we suppose that (3.2) has the formal travelling wave solution

$$U(\xi) = a_0 + \frac{a_1\phi + b_1\psi}{\mu_1\phi + \mu_2\psi + 1}, \quad V(\xi) = A_0 + \frac{A_1\phi + B_1\psi}{\mu_1\phi + \mu_2\psi + 1}, \tag{3.3}$$

where $\mu_1$, $\mu_2$, $a_0$, $a_1$, $b_1$, $A_0$, $A_1$ and $B_1$ are constants to be determined later, and the new variables $\phi$ and $\psi$ satisfy (2.5).

With the aid of Maple, substituting (3.3) along with (2.5) into (3.2) and setting the coefficients of the terms $\phi^i\psi^j$ to zero yields a set of over-determined algebraic equations with respect to $a_0$, $a_1$, $b_1$, $A_0$, $A_1$, $B_1$, $\mu_1$, $\mu_2$, $k$, $l$ and $\lambda$.

By use of the Maple soft package "Charsets" by Dongming Wang, which is based on the Wu-elimination method, solving the over-determined algebraic equations, we get the following results:

$$a_1 = -\frac{b_1 h_3}{h_1}, \ A_1 = \pm\frac{\sqrt{-\alpha}b_1 h_3}{\alpha h_1}, \ B_1 = \pm\frac{\sqrt{-\alpha}b_1}{\alpha}, \quad \mu_1 = -\frac{h_2\mu_2}{h_4}, \ \lambda = A_0\alpha + a_0\sqrt{-\alpha}, \ l = \pm\sqrt{-\alpha}, \tag{3.4}$$

where $a_0$, $A_0$, $b_1$, $\mu_2$ and $k$ are arbitrary constants.

According to (3.3)–(3.4) and the general solutions of (2.5) listed in the Appendix, we will obtain the following exact solutions for the (2+1)-dimensional Burgers equation.

Note that: Since the solutions obtained here are so many, we just list some new solutions for the (2+1)-dimensional Burgers equation to illustrate the efficiency of our method.

*Family 1.* When $h_1 = 1$, $h_2 = -1$ and $h_3 = h_4 = \pm 1$, then we can get a combination of tanh and tan function solutions:

$$u_1 = a_0 \mp \frac{b_1\tanh(\xi)}{\pm\mu_2\tanh(\xi) + \mu_2\tan(\xi) + 1} + \frac{b_1\tan(\xi)}{\pm\mu_2\tanh(\xi) + \mu_2\tan(\xi) + 1}, \quad v_1 = A_0 \pm \frac{\sqrt{-\alpha}b_1\tanh(\xi)}{\alpha(\pm\mu_2\tanh(\xi) + \mu_2\tan(\xi) + 1)} - \frac{\sqrt{-\alpha}b_1\tan(\xi)}{\alpha(\pm\mu_2\tanh(\xi) + \mu_2\tan(\xi) + 1)}, \tag{3.5}$$

where $\xi = k(x + ly + \lambda t)$, $a_0$, $A_0$, $b_1$, $k$, $l$ and $\lambda$ are arbitrary constants.

*Family 2.* When $h_1 = 1$, $h_2 = -1$ and $h_3 = h_4 = \pm\frac{1}{2}$, then we can get a combination of tanh, sec and tan function solutions:

$$u_2 = a_0 \mp \frac{b_1\tanh(\xi)}{2(\pm 2\mu_2\tanh(\xi) + \mu_2(\sec(\xi)\pm\tan(\xi)) + 1)} + \frac{b_1(\sec(\xi)\pm\tan(\xi))}{\pm 2\mu_2\tanh(\xi) + \mu_2(\sec(\xi)\pm\tan(\xi)) + 1}, \quad v_2 = A_0 \pm \frac{\sqrt{-\alpha}b_1\tanh(\xi)}{2\alpha(\pm 2\mu_2\tanh(\xi) + \mu_2(\sec(\xi)\pm\tan(\xi)) + 1)} - \frac{\sqrt{-\alpha}b_1(\sec(\xi)\pm\tan(\xi))}{\alpha(\pm 2\mu_2\tanh(\xi) + \mu_2(\sec(\xi)\pm\tan(\xi)) + 1)}, \tag{3.6}$$

where $\xi = k(x + ly + \lambda t)$, $a_0$, $A_0$, $b_1$, $k$, $l$ and $\lambda$ are arbitrary constants.

*Family 3.* When $h_1 = 1$, $h_2 = -1$, $h_3 = 0$ and $h_4 \neq 0$, then we can get a combination of tanh and rational function solutions:

$$u_3 = a_0 - \frac{b_1 h_4}{\mu_2\tanh(\xi)(h_4\xi + c_0) - \mu_2 h_4 + h_4(h_4\xi + c_0)}, \quad v_3 = A_0 + \frac{h_4\sqrt{-\alpha}b_1}{\alpha(\mu_2\tanh(\xi)(h_4\xi + c_0) + \mu_2 h_4 + h_4(h_4\xi + c_0))}, \tag{3.7}$$

where $\xi = k(x + ly + \lambda t)$, $h_4 \neq 0$, $c_0$, $a_0$, $A_0$, $b_1$, $k$, $l$ and $\lambda$ are arbitrary constants.

*Family 4.* When $h_1 = -\frac{1}{2}$, $h_2 = \frac{1}{2}$ and $h_3 = h_4 = \pm 1$, then we can get a combination of tanh, sech and tan function solutions:

$$u_4 = a_0 \pm \frac{4b_1(\tanh(\xi)\pm \mathrm{i}\,\mathrm{sech}(\xi))}{(\mp\mu_2(\tanh(\xi)\pm \mathrm{i}\,\mathrm{sech}(\xi)) + 2\mu_2\tan(\xi) + 2)} + \frac{2b_1\tan(\xi)}{(\mp\mu_2(\tanh(\xi)\pm \mathrm{i}\,\mathrm{sech}(\xi)) + 2\mu_2\tan(\xi) + 2)}, \quad v_4 = A_0 \mp \frac{4\sqrt{-\alpha}b_1(\tanh(\xi)\pm \mathrm{i}\,\mathrm{sech}(\xi))}{\alpha(\mp\mu_2(\tanh(\xi)\pm \mathrm{i}\,\mathrm{sech}(\xi)) + 2\mu_2\tan(\xi) + 2)} - \frac{2\sqrt{-\alpha}b_1\tan(\xi)}{\alpha(\mp\mu_2(\tanh(\xi)\pm \mathrm{i}\,\mathrm{sech}(\xi)) + 2\mu_2\tan(\xi) + 2)}, \tag{3.8}$$

where $\xi = k(x+ly+\lambda t)$, $a_0$, $A_0$, $b_1$, $k$, $l$ and $\lambda$ are arbitrary constants.

*Family 5.* When $h_1 = -\frac{1}{2}$, $h_2 = \frac{1}{2}$ and $h_3 = h_4 = \pm\frac{1}{2}$, then we can get a combination of tanh, sech, sec and tan function solutions:

$$\begin{aligned} u_5 &= a_0 \pm \frac{b_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi) \pm \sec(\xi) \pm \tan(\xi))}{\pm\mu_2(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1}, \\ v_3 &= A_0 \pm \frac{b_1\sqrt{-\alpha}(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi) \pm \sec(\xi) \pm \tan(\xi))}{\alpha(\pm\mu_2(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\sec(\xi) \pm \tan(\xi)) + 1)}, \end{aligned} \tag{3.9}$$

where $\xi = k(x+ly+\lambda t)$, $a_0$, $A_0$, $b_1$, $k$, $l$ and $\lambda$ are arbitrary constants.

*Family 6.* When $h_1 = 1$, $h_2 = -1$, $h_3 = 0$ and $h_4 \neq 0$, then we can get a combination of tanh, sech, rational function solutions:

$$\begin{aligned} u_6 &= a_0 - \frac{2b_1h_4}{-\mu_2(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))(h_4\xi + c_0) - 2\mu_2h_4 + 2h_4(h_4\xi + c_0)}, \\ v_6 &= A_0 + \frac{2\sqrt{-\alpha}b_1h_4}{\alpha(-\mu_2(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi))(h_4\xi + c_0) - 2\mu_2h_4 + 2h_4(h_4\xi + c_0))}, \end{aligned} \tag{3.10}$$

where $\xi = k(x+ly+\lambda t)$, $h_4 \neq 0$, $c_0$, $a_0$, $A_0$, $b_1$, $k$, $l$ and $\lambda$ are arbitrary constants.

*Family 7.* When $h_1 = h_2 = \pm 1$, $h_3 = 0$ and $h_4 \neq 0$, then we can get a combination of tan and rational function solutions:

$$\begin{aligned} u_7 &= a_0 - \frac{b_1h_4}{\mp\mu_2\tan(\xi)(h_4\xi + c_0) - h_4\mu_2 + h_4(h_4\xi + c_0)}, \\ v_7 &= A_0 + \frac{\sqrt{-\alpha}b_1h_4}{\alpha(\mp\mu_2\tan(\xi)(h_4\xi + c_0) - h_4\mu_2 + h_4(h_4\xi + c_0))}, \end{aligned} \tag{3.11}$$

where $\xi = k(x+ly+\lambda t)$, $h_4 \neq 0$, $c_0$, $a_0$, $A_0$, $b_1$, $k$, $l$ and $\lambda$ are arbitrary constants.

*Family 8.* When $h_1 = 1$, $h_2 = -1$, $h_3 = 0$ and $h_4 \neq 0$, then we can get a combination of tan, sec, rational function solutions:

$$\begin{aligned} u_8 &= a_0 - \frac{b_1h_4}{\mu_2(\sec(\xi) \pm \tan(\xi))(h_4\xi + c_0) - \mu_2h_4 + h_4(h_4\xi + c_0)}, \\ v_8 &= A_0 + \frac{\sqrt{-\alpha}b_1h_4}{\alpha(\mu_2(\sec(\xi) \pm \tan(\xi))(h_4\xi + c_0) - \mu_2h_4 + h_4(h_4\xi + c_0))}, \end{aligned} \tag{3.12}$$

where $\xi = k(x+ly+\lambda t)$, $h_4 \neq 0$, $c_0$, $a_0$, $A_0$, $b_1$, $k$, $l$ and $\lambda$ are arbitrary constants.

At the same time, we can also get some new solutions which are not the complexiton solutions and can not be obtained by another tanh method, such as:

*Family 9.* When $h_1 = h_3 = \frac{1}{2}$ and $h_2 = h_4 = -\frac{1}{2}$, then we obtain the following solutions:

$$\begin{aligned} u_9 &= a_0 \pm \frac{b_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi) \pm \coth(\xi) \pm \mathrm{csch}(\xi))}{-\mu_2(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\coth(\xi) \pm \mathrm{csch}(\xi)) + 1}, \\ v_9 &= A_0 \pm \frac{\sqrt{-\alpha}b_1(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi) \pm \coth(\xi) \pm \mathrm{csch}(\xi))}{\alpha(-\mu_2(\tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi)) + \mu_2(\coth(\xi) \pm \mathrm{csch}(\xi)) + 1)}, \end{aligned} \tag{3.13}$$

where $\xi = k(x+ly+\lambda t)$, $a_0$, $A_0$, $b_1$, $k$, $l$ and $\lambda$ are arbitrary constants.

*Family 10.* When $h_1 = h_2 = \pm\frac{1}{2}$ and $h_3 = h_4 = \pm\frac{1}{2}$, then we obtain the following solutions:

$$\begin{aligned} u_{10} &= a_0 \pm \frac{b_1(\sec(\xi) \pm \tan(\xi) \pm \csc(\xi) \pm \cot(\xi))}{\pm\mu_2(\sec(\xi) \pm \tan(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1}, \\ v_{10} &= A_0 \pm \frac{b_1\sqrt{-\alpha}(\sec(\xi) \pm \tan(\xi) \pm \csc(\xi) \pm \cot(\xi))}{\alpha(\pm\mu_2(\sec(\xi) \pm \tan(\xi)) + \mu_2(\csc(\xi) \pm \cot(\xi)) + 1)}, \end{aligned} \tag{3.14}$$

where $\xi = k(x+ly+\lambda t)$, $a_0$, $A_0$, $b_1$, $k$, $l$ and $\lambda$ are arbitrary constants.

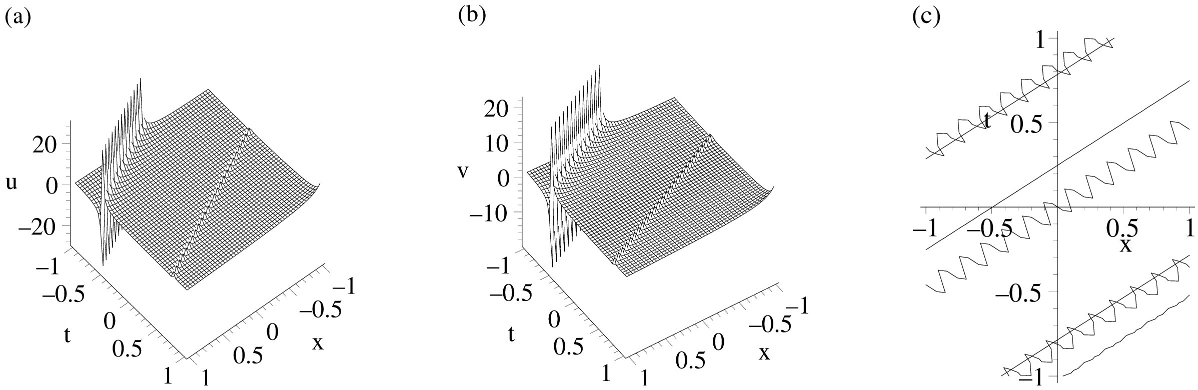


Fig. 1. Combination of the tanh and tan function solution of the (2+1)-dimensional Burgers fields $u_1$ (a) and $v_1$ (b), where $h_1 = h_3 = h_4 = b_1 = \mu_2 = k = 1$, $a_0 = -\frac{\sqrt{2}}{2}$, $A_0 = 2$, $h_2 = -1$, $\alpha = -2$ and $y = 0$. (c) Distribution of singular points of solutions $u_1$ and $v_1$.

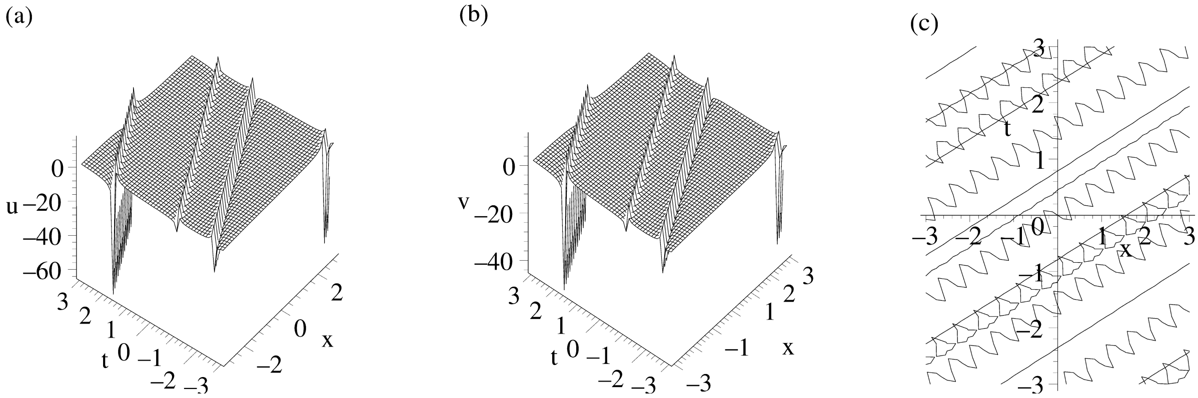


Fig. 2. Combination of tanh, sec and tan function solution of the (2+1)-dimensional Burgers fields $u_2$ (a) and $v_2$ (b), where $h_1 = b_1 = \mu_2 = k = 1$, $h_2 = -1$, $h_3 = h_4 = \frac{1}{2}$, $a_0 = -\frac{\sqrt{2}}{2}$, $A_0 = 2$, $\alpha = -2$ and $y = 0$. (c) Distribution of singular points of solutions $u_2$ and $v_2$.

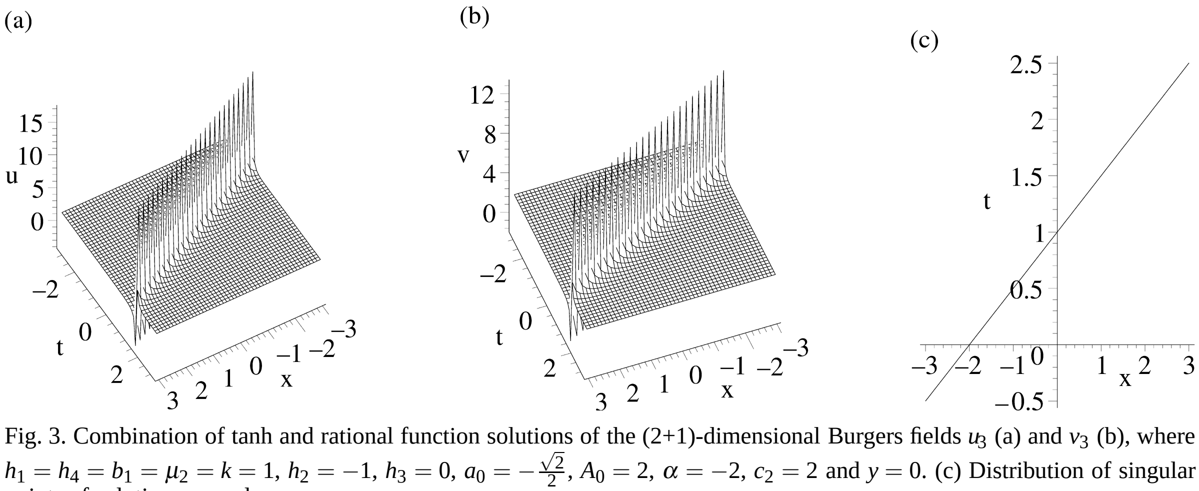


Fig. 3. Combination of tanh and rational function solutions of the (2+1)-dimensional Burgers fields $u_3$ (a) and $v_3$ (b), where $h_1 = h_4 = b_1 = \mu_2 = k = 1$, $h_2 = -1$, $h_3 = 0$, $a_0 = -\frac{\sqrt{2}}{2}$, $A_0 = 2$, $\alpha = -2$, $c_2 = 2$ and $y = 0$. (c) Distribution of singular points of solutions $u_3$ and $v_3$.

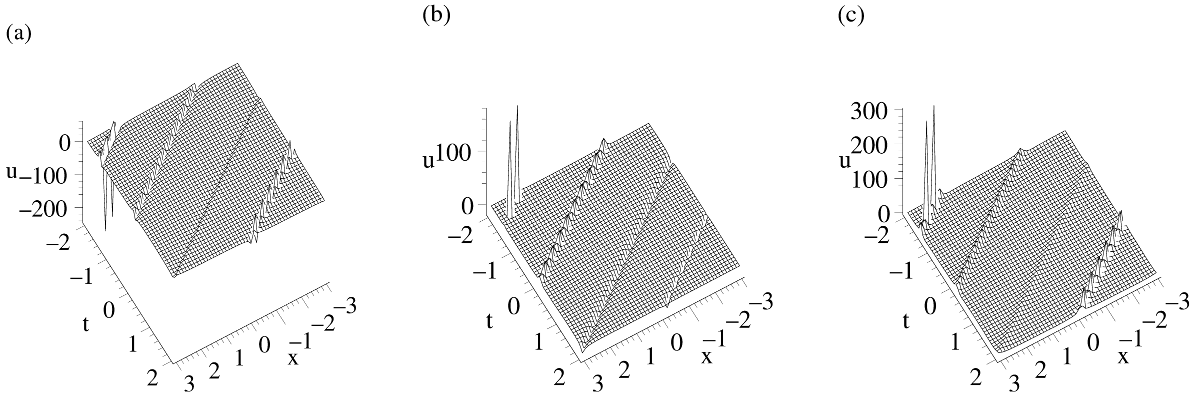


Fig. 4. Real part (a), imaginary part (b) and the modulus (c) of combinations of tanh, sech and tan solutions of the (2+1)-dimensional Burgers fields $u_4$, where $h_1 = h_4 = b_1 = \mu_2 = k = 1$, $h_2 = -1$, $h_3 = 0$, $a_0 = -\sqrt{2}$, $A_0 = 2$, $\alpha = -2$, $c_2 = 2$ and $y = 0$.

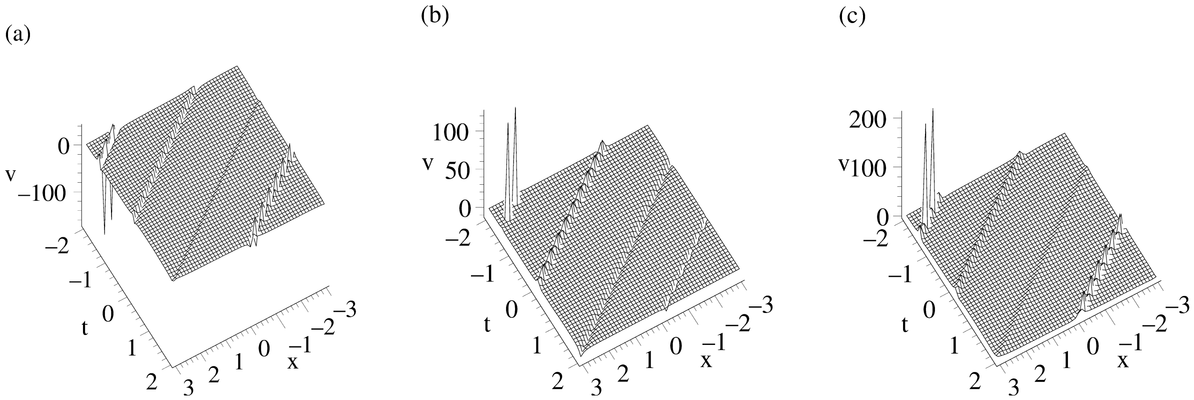


Fig. 5. Real part (a), imaginary part (b) and the modulus (c) of combinations of tanh, sech and tan solution of the (2+1)-dimensional Burgers fields $v_4$, where $h_1 = h_4 = b_1 = \mu_2 = k = 1$, $h_2 = -1$, $h_3 = 0$, $a_0 = -\frac{\sqrt{2}}{2}$, $A_0 = 2$, $\alpha = -2$, $c_2 = 2$ and $y = 0$.

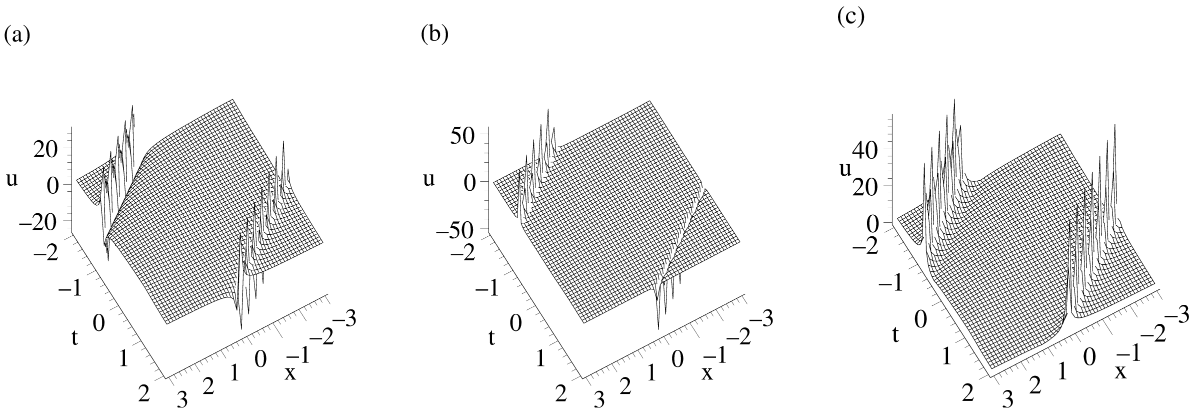


Fig. 6. Real part (a), imaginary part (b) and the modulus (c) of combinations of the tanh, sech, sec and tan function solution of the (2+1)-dimensional Burgers field $u_5$, where $h_1 = h_4 = b_1 = \mu_2 = k = 1$, $h_2 = -1$, $h_3 = 0$, $a_0 = -\frac{\sqrt{2}}{2}$, $A_0 = 2$, $\alpha = -2$, $c_2 = 2$ and $y = 0$.

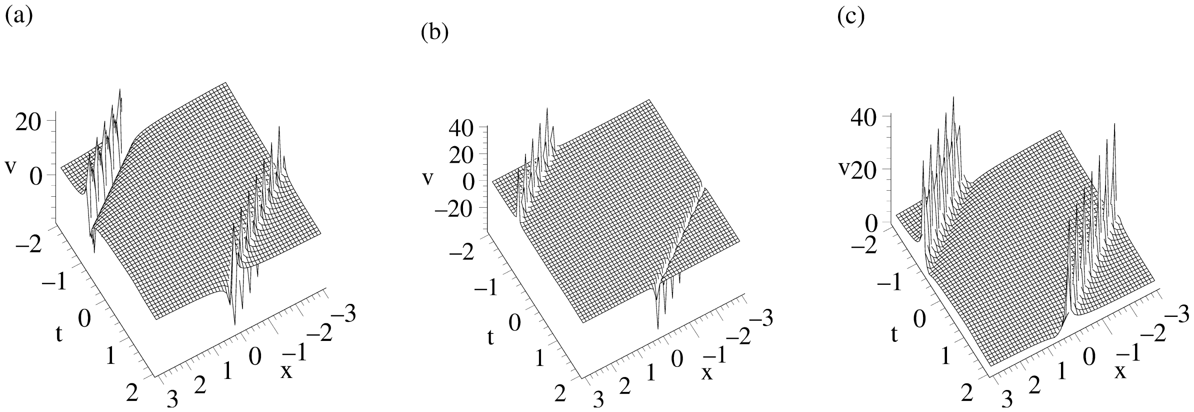


Fig. 7. Real part (a), imaginary part (b) and the modulus (c) of combinations of tanh, sech, sec and tan function solutions of the (2+1)-dimensional Burgers field $v_5$, where $h_1 = h_4 = b_1 = \mu_2 = k = 1$, $h_2 = -1$, $h_3 = 0$, $a_0 = -\frac{\sqrt{2}}{2}$, $A_0 = 2$, $\alpha = -2$, $c_2 = 2$ and $y = 0$.

The properties of these complexiton solutions: the combination of tanh and tan function solution; the combination of tanh, sec and tan function solution; the combination of tanh and rational function solution; the combination of tanh, sech and tan solution; the combination of tanh, sech, sec and tan function solution are shown by some figures (Figs. 1 – 7). In addition, Figs. 1c, 2c, 3c show the distribution of singular points of solutions $u_i$ and $v_i$ $(i = 1,2,3)$. For similarity, other corresponding distribution of singular points of other solutions are omitted.

## 4. Conclusion

Many new types of complexiton solutions to the (2+1)-dimensional Burgers equation have been investigated. These complexiton solutions possess various combinations of trigonometric periodic and hyperbolic function solutions, various combinations of trigonometric periodic and rational function solutions, various combinations of hyperbolic and rational function solutions. It is hoped that the study of these complexiton solutions could further assist understanding, identifying and classifying nonlinear integrable and non-integrable differential equations and their exact solutions. In fact, we naturally use two or more really different subequations to replace two Riccati equations in the multi-variable Riccati equation rational expansion method, such as the pair of Riccati equations and the elliptic equation [9]; thus we can obtain more rich complexiton solutions possessing various combinations of trigonometric periodic and elliptic function solutions, various combinations of elliptic function solutions and rational function solutions, various combinations of hyperbolic and rational function solutions, etc. These will be further considered.

## Appendix

The general solutions of the Riccati equation

$$\frac{dF}{d\xi} = R_1 + R_2 F^2$$

are:

1.) when $R_1 = \frac{1}{2}$ and $R_2 = -\frac{1}{2}$,

$$F(\xi) = \tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi),$$
$$F(\xi) = \coth(\xi) \pm \mathrm{csch}(\xi),$$

2.) when $R_1 = R_2 = \pm\frac{1}{2}$,

$$F(\xi) = \sec(\xi) \pm \tan(\xi),$$
$$F(\xi) = \csc(\xi) \pm \cot(\xi),$$

3.) when $R_1 = 1$ and $R_2 = -1$,

$$F(\xi) = \tanh(\xi), \quad F(\xi) = \coth(\xi),$$

4.) when $R_1 = R_2 = 1$,

$$F(\xi) = \tan(\xi),$$

5.) when $R_1 = R_2 = -1$,

$$F(\xi) = \cot(\xi),$$

6.) when $R_1 = 0$ and $R_2 \neq 0$,

$$F(\xi) = -\frac{1}{h_2\xi + c_0}.$$

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[1] E. G. Fan, Phys. Lett. A **277**, 212 (2000); E. G. Fan, J. Phys. A **36**, 7009 (2003).
[2] Z. Y. Yan, Comput. Math. Appl. **46**, 1363 (2003); Z. Y. Yan, Phys. Lett. A **292**, 100 (2001).
[3] Y. Chen and Y. Zheng, Int. J. Mod. Phys. C **14**, 601 (2003); B. Li, Y. Chen, and H. Q. Zhang, Z. Naturforsch. **57a**, 874 (2002).
[4] Y. Chen and Q. Wang, Z. Naturforsch. **60a**, 1 (2005).
[5] Q. Wang and Y. Chen, Commun. Theor. Phys. **41**, 821 (2004).
[6] W. X. Ma, Phys. Lett. **301**, 35 (2002).
[7] S. Y. Lou, H. C. Hu, and X. Y. Tang, Phys. Rev. E **71**, 036604 (2005).
[8] X. Y. Tang and S. Y. Lou, Chin. Phys. Lett. **20**, 335 (2003).
[9] Y. Chen, Q. Wang, and B. Li, Chaos, Solitons and Fractals **26**, 231 (2005).